

# NON- $G$ -COMPLETELY REDUCIBLE SUBGROUPS OF THE EXCEPTIONAL ALGEBRAIC GROUPS

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**ABSTRACT.** Let  $G$  be an exceptional algebraic group defined over an algebraically closed field  $k$  of characteristic  $p > 0$  and let  $H$  be a subgroup of  $G$ . Then following Serre we say  $H$  is  $G$ -completely reducible or  $G$ -cr if, whenever  $H$  is contained in a parabolic subgroup  $P$  of  $G$ , then  $H$  is in a Levi subgroup of that parabolic. Building on work of Liebeck and Seitz, we find all triples  $(X, G, p)$  such that there exists a closed, connected, simple non- $G$ -cr subgroup  $H \leq G$  with root system  $X$ .

## 1. INTRODUCTION

Let  $G$  be an algebraic group defined over an algebraically closed field  $k$  of characteristic  $p > 0$  and let  $H$  be a subgroup of  $G$ . Then following Serre [Ser98] we say  $H$  is  $G$ -completely reducible or  $G$ -cr if, whenever  $H$  is contained in a parabolic subgroup  $P$  of  $G$ , then  $H$  is in a Levi subgroup of that parabolic. This is a natural generalisation of the notion of a group acting completely reducibly on a module  $V$ : if we set  $G = GL(V)$  then saying  $H$  is  $G$ -completely reducible is precisely the same as saying that  $H$  acts semisimply on  $V$ .

This notion is important in unifying some other pre-existing notions and results. For instance, in [BMR05], it was shown that a subgroup  $H$  is  $G$ -cr if and only if it satisfied Richardson's notion of being strongly reductive in  $G$ . It also allows one to state some previous results due to Liebeck–Seitz and Liebeck–Saxl–Testerman on the subgroup structure of the exceptional algebraic groups in a particularly satisfying form:

Assume  $G$  is simple of one of the five exceptional types and let  $X$  be a simple root system. The result [LS96, Theorem 1] asserts a number  $N(X, G)$  such that if  $H$  is closed, connected and simple, with root system  $X$ , then  $H$  is  $G$ -cr whenever the characteristic  $p$  of  $k$  is bigger than  $N(X, G)$ . In particular if  $p$  is bigger than 7 then they show that all closed, connected, reductive subgroups of  $G$  are  $G$ -cr. There is some overlap in that paper with the contemporaneous work of [LST96]. If  $H$  is a simple subgroup of rank greater than half the rank of  $G$ , then [Theorem 1, *ibid.*] finds all conjugacy classes of simple subgroups of  $G$ , the proofs indicate where these conjugacy classes are  $G$ -completely reducible. With essentially one class of exceptions, all subgroups, including the non- $G$ -cr subgroups, can be located in ‘nice’ so-called subsystem subgroups of  $G$ . We shall mention these in greater detail later.

More recently, [Ste10a] and [Ste12] find all conjugacy classes of simple subgroups of exceptional groups of types  $G_2$  and  $F_4$ . One consequence of this is to show that the numbers  $N(X, G)$  found above can be made strict. (One need only change  $N(A_1, G_2)$  from 3 to 2.) The main purpose of this article is to make *all* the  $N(X, G)$  strict. That is, for each of the five types of exceptional algebraic group  $G$ , for each prime  $p = \text{char } k$  and for each simple root system  $X$ , we give in a table of Theorem 1 an example  $H = E(X, G, p)$  of a connected, closed, simple non- $G$ -cr subgroup  $H^1$  with

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<sup>1</sup>(thus,  $H$  is in some parabolic  $P$ , but in no Levi subgroup  $L$  of  $P$ )

root system  $X$ , precisely when this is possible. In other words we classify the triples  $(X, G, p)$  where there exists a connected, closed, simple non- $G$ -cr subgroup  $H$  with root system  $X$ . Moreover, in all but one case (where  $(X, G, p) = (G_2, E_7, 7)$ ), we can locate  $E(X, G, p)$  in a subsystem subgroup.

Our main theorem can thus be viewed as the best possible improvement of the result [LS96, Theorem 1], in the spirit of that result. Before we state our main theorem in full, we need a definition: A *subsystem subgroup* of  $G$  is a simple, closed, connected subgroup  $Y$  which is normalised by a maximal torus  $T$  of  $G$ . Let  $\Phi$  be the root system of  $G$  corresponding to a choice of Borel subgroup  $B \geq T$  and for  $\alpha \in \Phi$ , let  $U_\alpha$  denote the  $T$ -root subgroup corresponding to  $\alpha$ . Then  $Y = \langle U_\alpha | \alpha \in \Phi_0 \rangle$  where either  $\Phi_0$  is a closed subsystem of  $\Phi$  or  $(\Phi, p)$  is  $(B_n, 2)$ ,  $(C_n, 2)$ ,  $(F_4, 2)$  or  $(G_2, 3)$  and  $\Phi_0$  lies in the dual of a closed subsystem. The subsystem subgroups of  $G$  are easily determined by the Borel–de Siebenthal algorithm. Most of our examples  $H = E(X, G, p)$  are described in terms of an embedding of  $H$  into a subsystem subgroup  $M$ . Here we describe  $M$  just by giving its root system.

**Theorem 1.** *Let  $G$  be an exceptional algebraic group defined over an algebraically closed field  $k$  of characteristic  $p > 0$ . Suppose there exists a non- $G$ -cr closed, connected, simple subgroup  $H$  of  $G$  with root system  $X$ . Then  $(X, G, p)$  has an entry in Table 1.*

*Conversely, for each  $(X, G, p)$  given in Table 1, the last column guarantees an example of a closed, connected, simple, non- $G$ -cr subgroup  $E(X, G, p)$  with root system  $X$ .*

In particular we can improve on [LS96, Theorem 1]. In the table in Corollary 2 we have struck out the primes which were used in the hypotheses in [loc. cit.]. This is done partly to show where we have made improvements but mainly to facilitate reading the proof of the first part of Theorem 1.

**Corollary 2.** *Let  $G$  be an exceptional algebraic group over a field  $k$  of characteristic  $p$ . Let  $X$  be a simple root system and let  $N(X, G)$  be a list of primes defined by the table below. Suppose  $H$  is a closed, connected, reductive subgroup of  $G$  with root system having simple components  $X_1, \dots, X_n$ . Then if  $p \notin \bigcup_i N(X_i, G)$ ,  $H$  is  $G$ -cr.*

	$G = E_8$	$E_7$	$E_6$	$F_4$	$G_2$
$X = A_1$	$\leq 7$	$\leq 7$	$\leq 5$	$\leq 3$	$\cancel{3} \ 2$
$A_2$	$\cancel{5} \ 3 \ 2$	$\cancel{5} \ 3 \ 2$	$3 \ 2$	$3 \ \cancel{2}$	
$B_2$	$5 \ \cancel{3} \ 2$	$\cancel{3} \ 2$	$\cancel{3} \ 2$	$2$	
$G_2$	$7 \ \cancel{5} \ 3 \ 2$	$7 \ \cancel{5} \ 3 \ 2$	$\cancel{3} \ 2$	$2$	
$A_3$	$2$	$\cancel{2}$	$\cancel{2}$		
$B_3$	$2$	$2$	$2$	$2$	
$C_3$	$3 \ \cancel{2}$	$\cancel{2}$	$\cancel{2}$	$\cancel{2}$	
$B_4$	$2$	$\cancel{2}$	$\cancel{2}$		
$C_4, D_4$	$2$	$2$	$\cancel{2}$		

Using the above description of  $N(X, G)$  one also gets generalisations to each of the other results [LS96, Theorems 2–8], by replacing the hypothesis ‘ $p > N(X, G)$ ’ by ‘ $p \notin N(X, G)$ ’.

## 2. NOTATION

When discussing roots or weights, we use the Bourbaki conventions [Bou82, VI. Planches I–IX]. We use a lot of representation theory for algebraic groups whose notation we have taken largely consistent with [Jan03]. For an algebraic group  $G$ , recall that a  $G$ -module is a comodule for the Hopf algebra  $k[G]$ ; in particular every  $G$ -module is a  $kG$ -module. Let  $B$  be a Borel subgroup of

$G$	$X$	$p$	Example $E = E(X, G, p)$
$G_2$	$A_1$	2	$E \hookrightarrow A_1 A_1; x \mapsto (x, x)$
$F_4$	$A_1$	2	$E \hookrightarrow A_1^2; x \mapsto (x, x)$
		3	$E \hookrightarrow A_2^2; (V_3, V_3) \downarrow E = (2, 2)$
	$A_2$	3	$E \hookrightarrow A_2^2; x \mapsto (x, x)$
	$B_2$	2	$E \leq D_3$
	$G_2$	2	$E \leq D_4$
	$B_3$	2	$E \leq D_4$
$E_6$	$A_1$	2	$E \hookrightarrow A_1^2; x \mapsto (x, x)$
		3	$E \hookrightarrow A_2^2; (V_3, V_3) \downarrow E = (2, 2)$
		5	$E \hookrightarrow D_5; V_{10} \downarrow E = T(8)$
	$A_2$	2	$E \hookrightarrow A_5; V_6 \downarrow E = V(20) = 10^{[1]}/01$
		3	$E \hookrightarrow A_2^3; x \mapsto (x, x, x)$
	$B_2$	2	$E \hookrightarrow A_4; V_5 \downarrow E = V(10) = 10/00$
	$G_2$	2	$E \hookrightarrow C_4; V_8 \downarrow E = T(10)$
	$B_3$	2	$E \hookrightarrow C_4; V_8 \downarrow E = T(100)$
$E_7$	$E \leq E_6$	2, 3, 5	each of the subgroups of $E_6$ above
	$A_1$	7	$E \leq A_7; V_8 \downarrow E = W(7) = 1^{[1]}/5$
	$G_2$	7	$E$ in an $E_6$ -parabolic of $G$ *
	$C_4$	2	$E \hookrightarrow A_7; V_8 \downarrow E = L(1000)$
	$D_4$	2	$E \leq C_4$ above
$E_8$	$E \leq E_7$	2, 3, 5, 7	each of the subgroups of $E_7$ above
	$B_2$	5	$E \leq D_8; V_{16} \downarrow E = T(20) = 00/20/00$
	$A_3$	2	$E \leq D_8; V_{16} \downarrow E = T(101) = 000/101/000$
	$C_3$	3	$E \leq D_8; V_{16} \downarrow E = 000/010/000 + 000?$
	$B_4$	2	$E \leq A_8; V_9 \downarrow E = 1000/0000$

TABLE 1. Simple non- $G$ -cr subgroups of type  $X$  in the exceptional groups

a reductive algebraic group  $G$ , containing a maximal torus  $T$  of  $G$ . Recall that for each dominant weight  $\lambda \in X^+(T)$  for  $G$ , the space  $H^0(\lambda) := H^0(G/B, \lambda) = \text{Ind}_B^G(\lambda)$  is a  $G$ -module with highest weight  $\lambda$  and with socle  $\text{Soc}_G H^0(\lambda) = L(\lambda)$ , the irreducible  $G$ -module of highest weight  $\lambda$ . The Weyl module of highest weight  $\lambda$  is  $V(\lambda) \cong H^0(-w_0\lambda)^*$  where  $w_0$  is the longest element in the Weyl group. We identify  $X(T)$  with  $\mathbb{Z}^r$  for  $r$  the rank of  $G$  and for  $\lambda \in X(T)^+ \cong \mathbb{Z}_{\geq 0}^r \leq X(T)$ , write  $\lambda = (a_1, a_2, \dots, a_r) = a_1\omega_1 + \dots + a_r\omega_r$  where  $\omega_i$  are the fundamental dominant weights; a  $\mathbb{Z}_{\geq 0}$ -basis of  $X(T)^+$ . Put also  $L(\lambda) = L(a_1, a_2, \dots, a_r)$ . When  $0 \leq a_i < p$  for all  $i$ , we say that  $\lambda$  is a restricted weight and we write  $\lambda \in X_1(T)$ . Recall that any module  $V$  has a Frobenius twist  $V^{[n]}$  induced by raising entries of matrices in  $GL(V)$  to the  $p^n$ th power. Steinberg's tensor product theorem states that  $L(\lambda) = L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \dots \otimes L(\lambda_n)^{[n]}$  where  $\lambda_i \in X_1(T)$  and  $\lambda = \lambda_0 + p\lambda_1 + \dots + p^n\lambda_n$  is the  $p$ -adic expansion of  $\lambda \in \mathbb{Z}_+^r$ . We refer to  $\lambda_0$  as the restricted part of  $\lambda$ .

The right derived functors of  $\text{Hom}(V, *)$  are denoted by  $\text{Ext}_G^i(V, *)$  and when  $V = k$ , the trivial  $G$ -module, we have the identity  $\text{Ext}_G^i(k, *) = H^i(G, *)$  giving the Hochschild cohomology groups.

We recall some standard modules; when  $G$  is classical, there is a 'natural module' which we refer to by  $V_{\text{nat}}$ ; or  $V_m$  where  $m$  is the dimension of  $V_{\text{nat}}$ . It is always the Weyl module  $V(\omega_1)$ , which

is irreducible unless  $p = 2$  and  $G$  is of type  $B_n$ ; in the latter case it has a 1-dimensional radical. Certain properties of these modules is described in [Jan03, 8.21]. Of importance to us is the fact that when  $G = SL_n$ ,  $\bigwedge^r(L(\omega_1)) = L(\omega_r)$  for  $r \leq n - 1$ . We use this fact without further reference.

Recall that  $F_4$  has a 26-dimensional Weyl module which we denote ' $V_{26}$ '. When  $p \neq 3$ ,  $V_{26}$  is the irreducible representation of high weight  $0001 = \omega_4$ . When  $p = 3$ ,  $V_{26}$  has a one-dimensional radical, with a 25-dimensional irreducible quotient of high weight  $0001$ .  $E_6$  (resp.  $E_7$ ,  $E_8$ ) has a module of dimension 27 (resp. 56, 248) of high weight  $\omega_1$  (resp.  $\omega_7$ ,  $\omega_8$ ) which is irreducible in all characteristics. We refer to this module as  $V_{27}$  (resp.  $V_{56}$ ,  $\text{Lie}(E_8)$ ).

We will often want to consider restrictions of simple  $G$ -modules to reductive subgroups  $H$  of  $G$ . Where we write  $V_1|V_2|\dots|V_n$  we list the composition factors  $V_i$  of an  $H$ -module. For a direct sum of  $H$ -modules, we write  $V_1 + V_2$ . Where a module is uniserial, we will write  $V_1/\dots/V_n$  to indicate the socle and radical series: here the head is  $V_1$  and the socle  $V_n$ . On rare occasions we use  $V/W$  to indicate a quotient. It will be clear from the context which is being discussed.

Recall also the notion of a tilting module as one having a filtration by modules  $V(\mu)$  for various  $\mu$  and also a filtration by modules  $H^0(\mu)$  for various  $\mu$  (equiv. dual Weyl modules). Let us record in a lemma some key properties of tilting modules which we use:

**Lemma 2.1.** (i) *For each  $\lambda \in X(T)^+$  there is a unique indecomposable tilting module  $T(\lambda)$  of high weight  $\lambda$ ;*  
(ii) *A direct summand of a tilting module is a tilting module;*  
(iii) *The tensor product of two tilting modules is a tilting module;*  
(iv)  $\text{Ext}_G^1(T(\lambda), T(\mu)) = 0$ ; in particular  $H^1(G, T(\lambda)) = 0$ .

*Proof.* For (i), see [Don93, 1.1(i)]; (ii) is clear by projecting a filtration to a direct summand and using the fact that Weyl modules and induced modules are indecomposable; (iii) is [Don93, 1.2]; (iv) follows from [Jan03, II.4.13 (2)].  $\square$

As we are considering very low weight representations in general, it is possible to spot that a module is a  $T(\lambda)$ ; for instance when  $p = 2$ , the natural Weyl module for  $B_n$  has a 1-dimensional radical, so its structure is  $W(\lambda_1) = L(\lambda_1)/k$ . It is then the case that giving the Loewy series for a module  $k/L(\lambda_1)/k$  uniquely characterises it as a tilting module  $T(\lambda_1)$ .

Recall that a parabolic subgroup  $P$  of  $G$  has a Levi decomposition,

$$1 \rightarrow Q \rightarrow P \rightarrow L \rightarrow 1$$

where  $Q$  is the unipotent radical of the  $P$ . Recall also  $L = L'Z(L)$  with  $L'$  being semisimple.

### 3. OUTLINE

Theorem 1 has two facets. The first proves that if  $p \notin N(X, G)$  for  $N(X, G)$  as defined in Corollary 2, then  $X$  is  $G$ -cr. The second proves the existence of the examples given in Table 1 and proves that they are non- $G$ -cr.

The proof of the first part runs along the same lines as that of [LS96, Theorem 1]: Assume  $H$  is a closed, connected, simple non- $G$ -cr subgroup of  $G$ . Then  $H$  is a subgroup of  $P = LQ$ ; let  $\bar{H}$  be its image in  $L'$ . Almost all the time,  $H \cap Q = \{1\}$  as group-schemes and so we have  $HQ = \bar{H}Q$  and  $H$  is a complement to  $Q$  in  $\bar{H}Q$ . Then the possibilities for  $H$  are parameterised by  $H^1(\bar{H}, Q)$ ;

in fact, in any case, the possibilities for  $H$  are parameterised by  $H^1(\bar{H}, Q^{[1]})$ . This is the content of [Ste10b, Lemma 3.6.1].

From [ABS90],  $Q$  has a filtration  $Q = Q_1 \geq Q_2 \geq Q_3 \dots$  with successive quotients being known (usually semisimple)  $L$ -modules. So if we have  $H^1(\bar{H}, (Q_i/Q_{i+1})^{[1]}) = 0$  for each  $i$ , then (by 4.4(ii))  $H^1(\bar{H}, Q^{[1]}) = 0$  and  $H$  is conjugate to  $\bar{H}$ .

Now, for an exceptional algebraic group  $G$  over  $k$  of characteristic  $p$  and a simple root system  $X$  we consider possible embeddings  $\bar{H} \leq L'$  where  $\bar{H}$  is an  $L'$ -irreducible subgroup (which can be determined using 4.8 and/or by working down through 4.9). The composition factors  $V$  of the restrictions of the  $L$ -modules  $Q_i/Q_{i+1}$  are investigated, and then conditions for the vanishing of  $H^1(\bar{H}, V)$  found, for all relevant  $V$ . (Usually the dimensions of the composition factors are too small to admit non-vanishing of  $H^1(\bar{H}, V)$ .)

With essentially one exception, one can reduce to the case where  $V$  is of the form  $L(\lambda) \otimes L(\mu)^{[1]}$  with  $L(\lambda)$  non-trivial and restricted. There are any number of computer programs one can use to calculate the values of  $H^1(X, V)$  where  $\mu$  is 0.<sup>2</sup> Since the possible dimension of  $V$  is limited to a subset of roots of  $G$ , this process is finite.

For the proof of the second part of Theorem 1, we must show that for each of the remaining cases (where some composition factor  $V$  of  $Q^{[1]}$  has  $H^1(\bar{H}, V) \neq 0$ ), we exhibit a non- $G$ -cr subgroup  $H$  with the required root system over the required characteristic. In almost all cases we can give an example in a classical subgroup of  $G$ . Here it is easy to see when it is in a parabolic subgroup using 4.8. In two cases this is not possible, yet we can assert the existence of such a group using a cohomological argument.

#### 4. PRELIMINARIES

One needs to be careful about the notion of complements in semidirect products of algebraic groups. These are treated systematically in [McN10]. We recall some of the main facts.

**Definition 4.1** (cf. [McN10, 4.3.1]). Let  $G = H \ltimes Q$  be a semidirect product of algebraic groups as in [Jan03, I.2.6].

A closed subgroup  $H'$  of  $G$  is a *complement* to  $Q$  if it satisfies the following equivalent conditions:

- (i) Multiplication is an isomorphism  $H' \ltimes Q \rightarrow G$ .
- (ii)  $\pi_{H'} : H' \rightarrow H$  is an isomorphism of algebraic groups
- (iii) As group-schemes,  $H'Q = G$  and  $H' \cap Q = \{1\}$ .
- (iv) For the groups of  $k$ -points, one has  $H'(k)Q(k) = G(k)$ ,  $H'(k) \cap Q(k) = \{1\}$  and  $\text{Lie}(H') \cap \text{Lie}(Q) = 0$ .

*Remark 4.2.* See [Ste10b, §3.2] for a discussion. Note that [LS96] uses item (iv) above as its definition of a complement, without the last condition on Lie algebras.

**Definition 4.3.** A rational map  $\gamma : H \rightarrow Q$  is a 1-cocycle if  $\gamma(nm) = \gamma(n)^m \gamma(m)$  for each  $n, m \in N(k)$ . We write  $Z^1(H, Q)$  for the set of 1-cocycles.

We say  $\gamma \sim \delta$  if there is an element  $q \in Q(k)$  with  $q^{-h} \gamma(h) q = \delta(h)$  for each  $h \in H(k)$ . We write  $H^1(H, Q)$  for the set of equivalence classes of 1-cocycles  $Z^1(H, Q) / \sim$ .

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<sup>2</sup>We use the data on Frank Lübeck's website which accompanies [Lüb01].

We recall some results from [Ste10b].

- Lemma 4.4.** (i) *The set of 1-cocycles  $Z(H, Q)$  is in bijection with the set of complements to  $Q$  in  $HQ$ . Two cocycles are equivalent if the corresponding complements are conjugate by an element of  $H(k)$ .*
- (ii) *Suppose  $H$  is a closed, connected, reductive subgroup of a parabolic subgroup  $P = LQ$  of  $G$  and denote by  $\bar{H}$  the subgroup of  $L$  given by the image of  $H$  under the quotient map  $\pi : P \rightarrow L$ .*  
*Then as abstract groups  $H(k)$  is a complement to  $Q(k)$  in  $\bar{H}(k)Q(k)$ ; and either (1)  $H$  is a complement to  $Q$  in  $\bar{H}Q$ ; or (2)*
- (a)  $p = 2$ ;
  - (b) *There exists a component  $SO_{2n+1}$  of the semisimple group  $H/Z(H)^\circ$ ;*
  - (c) *the image of this component in  $\bar{H}/Z(\bar{H})^\circ$  is isomorphic to  $Sp_{2n}$ ; and*
  - (d) *the natural module for  $Sp_{2n}$  appears in a filtration of  $Q$  by  $\bar{H}$ -modules.*
- In case (2),  $H$  corresponds to a cocycle  $\gamma \in Z^1(\bar{H}, Q^{[1]})$  such that  $[\gamma]$  has no preimage in  $H^1(\bar{H}, Q)$  under the inclusion  $H^1(\bar{H}, Q) \rightarrow H^1(\bar{H}, Q^{[1]})$ .*
- Thus there is a bijection between the set of conjugacy classes of closed, connected, reductive subgroups  $H$  of  $\bar{H}Q$  and the set  $H^1(\bar{H}, Q^{[1]})$ .*
- (iii) *In a filtration of a unipotent algebraic  $H$ -group  $Q$  by  $H$ -modules (such as that given by 5.1) if each composition factor  $V$  satisfies  $H^1(H, V) = 0$  then  $H^1(H, Q) = 0$ .*

*Proof.* (i) is [Ste10b, Lemma 3.2.2]; (ii) is [Ste10b, Lemma 3.6.1]. For (iii), such a filtration is ‘sectioned’ in the sense of [Ste10b, Definition 3.2.7] using [Ste10b, Lemma 3.2.8]. Now one uses the exact sequence of non-abelian cohomology in [Ste10b, 2.1(i)] inductively. (See the discussion in [Ste10b, §3.2] on the validity of this sequence for rational cohomology.)  $\square$

In almost all cases the cohomology group  $H^1(G, V)$  for a semisimple algebraic group  $G$  satisfies  $H^1(G, V) \cong H^1(G, V^{[1]})$ . This fact allows us to reduce our considerations to simple modules with non-trivial restricted parts.

**Lemma 4.5.** *Let  $G$  be a simple algebraic group and  $V$  a simple  $G$ -module. Then  $H^1(G, V) \cong H^1(G, V^{[1]})$  unless  $G$  is  $Sp_{2n}$  and  $V$  is its  $2n$ -dimensional natural module.*

*Moreover  $H^1(G, V^{[1]})$  is isomorphic to its generic cohomology  $H_{gen}^1(G, V)$ .*

*Proof.* See [Jan03, II.12.2, Remark] and [CPSvdK77, 7.1].  $\square$

There are many papers finding the values  $\text{Ext}_H^n(L, M)$  with  $H$  of low rank and  $L, M$  simple. Taking  $L = k$ , the trivial module, one gets the following result, where we have included more data than necessary for our purposes for completion’s sake.

**Lemma 4.6.** *Let  $V$  be a simple module for a simple algebraic group  $H$  where  $H$  is one of  $SL_2$ ,  $SL_3$ ,  $Sp_4$  over an algebraically closed field of any characteristic  $p$ ;  $G_2$  for  $p = 2, 3$  or  $p \geq 13$ ; or  $SL_4$ ,  $Sp_6$  or  $Sp_8$  when  $p = 2$ . Then  $H^1(H, V)$  is at most one-dimensional, and is non-zero if and only if  $V$  is a Frobenius twist of one of the modules in the following table.*

*In the table we also give some useful dimension data, often in only specific characteristic.*

*Proof.* These are special cases from [Cli79] for  $SL_2$ , [Yeh82, 4.2.2] for  $SL_3$ , [Ye90] for  $Sp_4$ ,  $p \geq 3$ , [LY93] for  $G_2$  ( $p \geq 13$ ), [Sin94, Proposition 2.2] for  $Sp_4$  ( $p = 2$ ), [Sin94, Proposition 3.4] for  $G_2$



$H$	$p$	$L$	$\dim L$
$SL_2$	any	$L(p-2) \otimes L(1)^{[1]}$	$2p-2$
$SL_3$	$p \geq 3$	$L(p-2, p-2)$	$(p-1)^3 - 1$
		$L(1, p-2) \otimes L(1, 0)^{[1]}$	54 for $p = 5$
		$L(p-2, 1) \otimes L(0, 1)^{[1]}$	54 for $p = 5$
	$p = 2$	$L(1, 0) \otimes L(1, 0)^{[1]}$	9
		$L(0, 1) \otimes L(0, 1)^{[1]}$	9
$Sp_4$	$p \geq 5$	$L(0, p-3)$	
	$p \geq 3$	$L(2, p-2) \otimes L(0, 1)^{[1]}$	125 for $p = 3$
		$L(p-2, 1) \otimes L(1, 0)^{[1]}$	$\geq 64$
	$p = 2$	$L(1, 0)^{[1]}$	4
		$L(0, 1)$	4
$G_2$	$p \geq 13$	$L(p-5, 0)$	
		$L(p-2, 1) \otimes L(1, 0)^{[1]}$	
		$L(4, p-4) \otimes L(1, 0)^{[1]}$	
		$L(3, p-2)$	
		$L(3, p-2) \otimes L(0, 1)^{[1]}$	
	$p = 3$	$L(1, 1)$	49
		$L(0, 1) \otimes L(1, 0)^{[1]}$	49
	$p = 2$	$L(1, 0)$	6
		$L(0, 1) \otimes L(1, 0)^{[1]}$	84

$H$	$p$	$L$	$\dim L$
$SL_4$	$p = 2$	$L(1, 0, 1)$	14
		$L(0, 1, 0) \otimes L(1, 0, 0)^{[1]}$	24
		$L(0, 1, 0) \otimes L(0, 0, 1)^{[1]}$	24
		$L(1, 0, 1) \otimes L(0, 1, 0)^{[1]}$	84
$Sp_6$	$p = 2$	$L(1, 0, 0)^{[1]}$	6
		$L(1, 0, 1)$	48
		$L(0, 1, 0) \otimes L(1, 0, 0)^{[1]}$	84
$Sp_8$	$p = 2$	$L(1, 0, 0, 0)^{[1]}$	8
		$L(0, 1, 0, 0)$	26
		$L(1, 0, 1, 0)$	246
		$L(1, 0, 1, 0) \otimes L(0, 1, 0, 0)$	6396
		$L(1, 0, 1, 0) \otimes L(0, 1, 0, 0)$	6396
		$L(0, 1, 0, 1)$	416

( $p = 3$ ), [DS96, II.§2.1.6, II.§2.2.4, II.§3.3.6, III.§2.2.4] for  $SL_4$ ,  $Sp_6$ ,  $G_2$  and  $Sp_8$ , respectively, when  $p = 2$ .  $\square$

**Lemma 4.7.** *Let  $G = G_2$  over a field of characteristic 5 and let  $L$  be a simple module for  $G$  with  $H^1(G, L) \neq 0$ . Then  $\dim L > 56$ .*

*Proof.* One reduces to the case where the restricted part of  $L$  is non-trivial using 4.5 so we may assume  $L = M$ . Start with the case that  $M$  is restricted. One can use the data from [Lüb01] to establish that all Weyl modules of dimension less than 97 are irreducible. But then  $H^1(G, L(\lambda)) \cong H^0(G, H^0(\lambda)/\text{Soc}_G(H^0(\lambda))) = 0$ .

If  $M$  is not restricted, then it is  $M = M_1 \otimes M_2^{[1]}$  for  $M_1$  restricted and  $M_2$  non-trivial. The lowest dimension  $M_1$  and  $M_2$  can have is 7, the next is 14, but  $14 \times 7 > 56$ , so we conclude  $M_1 = L(1, 0)$  and  $M_2 = L(1, 0)^{[r]}$ . Now by [LS96, 1.15] (or the linkage principle), one gets  $H^1(G, M) = 0$ .  $\square$

The next lemma is useful for establishing  $L'$ -irreducible embeddings  $\bar{H} \leq L'$  when  $L'$  and also for deciding when a subgroup  $H$  is in a parabolic of a classical subgroup  $M$  of  $G$ .

**Lemma 4.8** ([LS96, p32-33]). *Let  $G$  be a simple algebraic group of classical type, with natural module  $V = V_G(\lambda_1)$ , and let  $H$  be a  $G$ -irreducible subgroup of  $G$ .*

- (i) *If  $G = A_n$ , then  $H$  acts irreducibly on  $V$*
- (ii) *If  $G = B_n, C_n$ , or  $D_n$  with  $p \neq 2$ , then  $V \downarrow H = V_1 \perp \cdots \perp V_k$  with the  $V_i$  all non-degenerate, irreducible, and inequivalent as  $X$ -modules.*
- (iii) *If  $G = D_n$  and  $p = 2$ , then  $V \downarrow H = V_1 \perp \cdots \perp V_k$  with the  $V_i$  all non-degenerate,  $V_2 \downarrow H, \dots, V_k \downarrow H$ , irreducible and inequivalent, and if  $V_1 \neq 0$ ,  $H$  acting on  $V_1$  as a  $B_{m-1}$ -irreducible subgroup where  $\dim V_1 = 2m$ .*

On a couple of occasions we need to know the reductive maximal subgroups of  $E_6$  and  $E_7$ .

**Lemma 4.9** (c.f. [LS04, Theorem 1]). *Let  $G$  be an exceptional group not of type  $E_8$  and let  $M$  be a closed, connected, reductive maximal subgroup of  $G$  without factors of  $A_1$  or connected centre. Then  $M$  is in the following list*

$G$	Subsystem $M$	Non-subsystem $M$
$G_2$	$A_2, A_2 (p = 3)$	
$F_4$	$B_4, C_4(p = 2), D_4, \tilde{D}_4 (p = 2), A_2\tilde{A}_2$	$G_2 (p = 7)$
$E_6$	$A_2^3$	$A_2 (p \neq 2, 3), G_2 (p \neq 7),$ $C_4 (p \neq 2), F_4, A_2G_2.$
$E_7$	$A_7, A_2A_5$	$A_2 (p \geq 5), G_2C_3$

## 5. PROOF OF THEOREM 1

In [Ste10a] and [Ste12] we find all semisimple non- $G$ -cr subgroups of  $G$  where  $G$  is  $G_2$  and  $F_4$  respectively. So the result follows for these cases. It remains to deal with the cases  $G = E_6, E_7$  and  $E_8$ . We start by honing the Liebeck and Seitz result to show that if  $H$  is a closed, connected, simple subgroup of  $G$  with root system  $X$  and  $p$  is not in our list  $N(X, G)$  then  $H$  is  $G$ -cr. Then we check that the examples given in Table 1 are indeed non- $G$ -cr.

A filtration for unipotent radicals of parabolics by  $L$ -modules is given in [ABS90]; to find the isomorphism types of the composition factors is a simple calculation using the root system of  $G$ . Summarising the results for our situation, we get:

**Lemma 5.1** ([LS96, 3.1]). *Let  $G = E_6, E_7$  or  $E_8$  and let  $P = LQ$  be a parabolic subgroup of  $G$ . The  $L'$ -composition factors within  $Q$  have the structure of high weight modules for  $L'$ . If  $L_0$  is a simple factor of  $L'$ , then the possible high weights  $\lambda$  of non-trivial  $L_0$ -composition factors and their dimensions are as follows:*

- (i)  $L_0 = A_n$ :  $\lambda = \lambda_j$  or  $\lambda_{n+1-j}$  ( $j = 1, 2, 3$ ), dimensions  $\binom{n+1}{j}$ ;
- (ii)  $L_0 = D_n$ :  $\lambda = \lambda_1, \lambda_{n-1}$  or  $\lambda_n$ , dimensions  $2n, 2^{n-1}$  and  $2^{n-1}$  resp.;
- (iii)  $L_0 = E_6$ :  $\lambda = \lambda_1$  or  $\lambda_6$ , dimension 27 each;
- (iv)  $L_0 = E_7$ :  $\lambda = \lambda_7$ , dimension 56.

**Corollary 5.2.** *With the hypotheses of the lemma, let  $V$  be an  $L'$ -composition factor of  $Q$  and suppose  $L'$  does not contain a component of type  $A_1$ . Then either  $\dim V \leq 60$  or  $G = E_8, L' = D_7$  and  $V$  is a spin module for  $L'$  of dimension 64.*

*If  $G = E_7, \dim V \leq 35$ ; if  $G = E_6, \dim V \leq 20$ .*

*Proof.* If  $L'$  is itself simple, this follows from the lemma. Also, if  $G = E_6$  or  $E_7$  then the number of positive roots is less than 56, so the result is clear. So we may assume  $G = E_8$ . The possibilities for  $L$  are  $A_2A_2, A_2A_3, A_2A_4, A_3A_3, A_3A_4, A_2D_4$  and  $A_2D_5$ . Since  $V$  is simple, it must be a tensor product of simple modules for the two factors, with the simple modules occurring in the lemma. One checks that the highest dimension possible for this is when  $L = A_3A_4, V = L(\lambda_2) \otimes L(\lambda_2)$  with  $\dim V = 6 \times 10 = 60$ .

For the second part, if  $G = E_7$  and  $L'$  is simple this follows from Lemma 5.1, the largest case occurring when  $L' = A_6$ . If  $L'$  is not simple, then it is  $A_4A_2, A_3A_2$  or  $A_2A_2$ . Then the largest possible dimension comes from the first option and is at most  $10 \times 3 = 30 \leq 35$ -dimensional.  $\square$



$p \notin N(X, G)$  **implies that  $H$  is  $G$ -cr.** Since we are building on [LS96, Theorem 1], we need only deal with the struck out numbers in the table in Corollary 2.

*Proof of the first statement of Theorem 1:*

Looking for a contradiction, we will assume  $H$  is non- $G$ -cr; then we can make the following assumption, using 4.4:

*We have  $H \leq P = LQ$  with  $\bar{H}$  being  $L$ -ir, and either (i)  $H$  is a complement to  $Q$  in  $\bar{H}Q$  and there exists a composition factor  $V$  of  $Q$  with  $H^1(\bar{H}, V) \neq 0$ ; or (ii)  $p = 2$ ,  $H = SO_{2n}$ ,  $\bar{H} = Sp_{2n}$  and  $V = L(\omega_1)$  appears as a composition factor of  $Q$ .*

The cases to consider are

$$(X, G, p) \in \{(B_2, \bullet, 3), (G_2, \bullet, 5), (G_2, E_6, 3), (A_2, \bullet, 5), (A_3, E_6, 2), \\ (A_3, E_7, 2), (B_4, E_6, 2), (B_4, E_7, 2), (D_4, E_6, 2), \\ (C_3, \bullet, 2), (C_4, \bullet, 2)\},$$

where  $\bullet$  can be replaced by  $E_6$ ,  $E_7$  or  $E_8$ .

By Corollary 5.2 the largest possibility for the dimension of  $V$  occurs when  $G = E_8$ ,  $L' = D_7$  and  $V$  has dimension 64. By 4.6, there is no such  $V$  when  $H = G_2$  and  $p = 5$ . This rules out  $(G_2, \bullet, 5)$ .

Suppose  $H$  is of type  $B_2$  and  $p = 3$ . Since  $\bar{H}$  is  $D_7$ -irreducible, it must have act on the natural module  $V_{14}$  for  $L'$  as specified in 4.8. Checking [Lüb01], one finds the simple untwisted representations of dimension no more than 14 are  $L(0, 1)$ ,  $L(1, 0)$ ,  $L(0, 2)$ ,  $L(2, 0)$  with dimensions 4, 5, 10 and 14, respectively. But  $L(0, 1)$  is the natural representation for  $Sp_4$ , thus carries a symplectic structure, which cannot be non-degenerate. Hence  $V_{14} \downarrow \bar{H} = L(2, 0)$ ; moreover, as  $L(2, 0)$  is an irreducible Weyl module when  $p = 3$ , the embedding  $\bar{H} \hookrightarrow L'$  can be seen as the reduction mod  $p$  of an embedding  $\bar{H}_{\mathbb{Z}} \hookrightarrow L'_{\mathbb{Z}}$ . Now [LS96, Proposition 2.12] gives that  $V_{\mathbb{Z}} \downarrow \bar{H}_{\mathbb{Z}}$  is the irreducible Weyl module  $V(1, 3)$ . Using [Lüb01] one can calculate the composition factors of a reduction mod 3 of this module; one sees that  $V \downarrow \bar{H}$  has composition factors  $L(1, 3)|L(2, 1)|L(0, 1)$ . Since none of these modules appears in 4.6, this rules out  $(X, G, p) = (B_2, \bullet, 3)$ .

By 5.2 the largest possibility for the dimension  $V$  when  $G = E_7$  is 35; when  $G = E_6$  it is 16.

Then dimension considerations using 4.6 and 4.7 also rule out  $(X, G, p) = (A_2, E_7, 5)$  and  $(G_2, E_6, 3)$ , respectively.

For  $(A_2, E_8, 5)$ , the fact that  $V$  has dimension at least 54 forces  $L' = E_7$ ,  $D_7$  or  $A_7$  but simple  $E_7$ - and  $D_7$ -modules are self-dual, so the possibilities for  $V$  coming from 4.6 are discounted as they are not self-dual. Thus we may assume that  $L' = A_7$  and  $V = L(\omega_3) = \bigwedge^3(L(\omega_1))$ . Since  $\bar{H}$  is  $L'$ -ir,  $L'$  must act irreducibly on the natural 8-dimensional module  $V_8$  for  $L'$ . A check of [Lüb01] forces  $V_8|L' = L(1, 1)$ . But  $\bigwedge^3 L(1, 1)$  has highest weights  $(2, 2)$  and  $(0, 3)$ . But the weights appearing in 4.6 are all higher than these (in the dominance order). This rules out  $(A_2, E_8, 5)$ .

Consider next the case  $(X, G, p) = (A_3, E_6, 2)$ . By 5.2 we have  $\dim V \leq 20$  so 4.6 shows that  $V$  must be 14-dimensional; this forces  $L' = D_5$  or  $A_5$ . Examining low dimensional representations for  $A_3$ , it is easy to see using 4.8 that there is no  $D_5$ -irreducible embedding  $\bar{H} \hookrightarrow D_5$ , so we must have  $\bar{H} \hookrightarrow L' = A_5$  by  $V_6|_{\bar{H}} = L(0, 1, 0)$ . Here,  $Q$  has factors  $L(\lambda_3) = \bigwedge^3(V_6)$  and a trivial module. Now  $L(0, 1, 0)$  has weights  $\pm(0, 1, 0), \pm(1, 0, -1), \pm(1, -1, 1)$ , so  $\bigwedge^3 L(0, 1, 0)$  has dominant weights  $(0, 0, 2)$ ,  $(2, 0, 0)$  and  $(0, 1, 0)$ . These do not appear in 4.6. Thus  $H^1(\bar{H}, \bigwedge^3 L(0, 1, 0)) = 0$  and this case is ruled out.

Let  $(X, G, p) = (A_3, E_7, 2)$ . Again  $V$  is at least 14-dimensional. So  $L' = A_5, A_6, D_5, D_6$  or  $E_6$ . Using 4.8 and 4.9 for  $L' = D_5$  and  $E_6$  respectively, one finds there are no  $L'$ -irreducible subgroups of type  $A_3$ . Thus  $L'$  is  $A_5$  or  $D_6$ ; a similar analysis to the case  $(A_3, E_6, 2)$  rules out the former as an option. So we have  $\bar{H} \leq A_3^2 \leq L' = D_6$  by  $V_{12} \downarrow \bar{H} = L(0, 1, 0) + L(0, 1, 0)^{[r]}$ . Now  $Q$  has  $L'$ -composition factors  $k$  and  $L(\omega_6)$ , a spin module. We wish to calculate  $L(\omega_6) \downarrow \bar{H}$ . Since  $\bar{H} \leq A_3^2$  it is instructive to work out  $L(\omega_6)$  restricted to one of these factors. Using [LS96, 2.6 and 2.7] this is  $L(1, 0, 0)^4 + L(0, 0, 1)^4$ . Thus we must have  $L(\omega_6) \downarrow A_3^2 = (L(1, 0, 0), L(1, 0, 0)) + (L(0, 0, 1), L(0, 0, 1))$  so that  $L(\omega_6) \downarrow \bar{H} = L(1, 0, 0) \otimes L(1, 0, 0)^{[1]} + L(0, 0, 1) \otimes L(0, 0, 1)^{[1]}$ .<sup>3</sup> Now 4.6 implies  $H^1(\bar{H}, Q) = 0$ .

In case  $(B_4, E_6, 2)$  we must have  $\bar{H} \leq D_5$ , with  $Q$  a spin module for  $L'$ . But then  $Q \downarrow \bar{H} = V \cong L(0001)$  using [LS96, 2.7] is a spin module for  $H$  with  $V(0001) = L(0001)$ . So  $H^1(B_4, V) = 0$  and this case is ruled out.

Lastly take case  $(X, G, p) = (C_3, \bullet, 2)$  of type  $C_3$ . We need an  $L'$ -ir embedding of  $\bar{H}$  in  $L'$  and an  $H$ -composition factor  $V$  of  $Q$  with  $H^1(H, V) \neq 0$ . We will see this is impossible. As above, if  $G = E_6$ ,  $L'$  has to be type  $A_5$ , with  $Q$  having  $L'$ -composition factors  $k$  and  $L(0, 0, 1, 0, 0) = \bigwedge^3 L(1, 0, 0, 0, 0)$ . Hence  $Q$  has  $\bar{H}$  composition factors which are  $k$  or in  $\bigwedge^3 L(1, 0, 0)$  which has composition factors  $L(0, 0, 1)|L(1, 0, 0)^2$ . Since these do not appear in 4.6 this case is ruled out. Similarly if  $G = E_7$  or  $E_8$  we must still have  $L' = A_5$  and we must also consider the restrictions of  $L(0, 1, 0, 0, 0)$  and its dual,  $L(0, 0, 0, 1, 0)$  to  $\bar{H}$ . These are  $\bigwedge^2 L(1, 0, 0) \cong \bigwedge^4 L(1, 0, 0)$  which also contain no composition factors with non-trivial  $H^1$ .

Since there are no embeddings of a subgroup of type  $C_4$  into any proper Levi of  $E_6$ , this case is ruled out too.

This completes the proof of the first statement of Theorem 1.

$p \in N(X, G)$  **implies the existence of a non- $G$ -cr subgroup  $H$  with root system  $X$ .** The examples when  $G = G_2$  and  $F_4$  were shown already in [Ste10a, Theorem 1] and [Ste12, Theorem 1(A)(B)] to be non- $G$ -cr, so we need only deal with the cases  $G = E_6, E_7$  and  $E_8$ .

*Proof of the second part of Theorem 1: The subgroups listed in Table 1 are non- $G$ -cr:*

The proof of many of these cases is similar. Let  $H = E(X, G, p)$  for one of the examples in Table 1. We locate  $H$  within a parabolic subgroup of  $G$  and establish the embedding  $\bar{H} \leq L$ . Next we take a low dimensional (faithful)  $G$ -module  $V$  and calculate the restriction to  $H$  and  $\bar{H}$  of this  $G$ -module; in all cases under consideration these will be non-isomorphic. Thus we can conclude that since  $V \downarrow H \not\cong V \downarrow \bar{H}$ ,  $H$  is not even  $GL(V)$ -conjugate to  $\bar{H}$ , let alone  $G$ -conjugate to  $\bar{H}$ . Further, in all the cases under consideration we will conclude that if  $H$  is non- $F_4$ -cr, then it is also non- $E_r$ -cr for  $6 \leq r \leq 8$  using the embeddings  $F_4 \leq E_6 \leq E_7 \leq E_8$ ; unfortunately we seem to need to do this mostly case by case.

We will now give a few examples.

$H = E(E_6, A_1, 2)$

Here  $H$  is a subgroup of type  $A_1$  in a subsystem  $A_1^2$  given by  $A_1 \hookrightarrow A_1^2$  by  $x \mapsto (x, x)$ . From [LS04, Table 10.1] we have  $V_{27} \downarrow F_4 = V_{26} + k$ . Now from [Ste12, 5.1] we have  $V_{26} \downarrow A_1^2 =$

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<sup>3</sup>This statement is made without loss of generality: one can embed with graph automorphisms to have dual versions of these modules.

$L(1,1) + k^6 + L(1,0)^4 + L(0,1)^4$ , so  $V_{27} \downarrow H = L(1) \otimes L(1) + L(1)^8 + k^7 = T(2) + L(1)^8 + k^7$ . In [Ste12] it is shown that  $H$  is in a long  $A_1$ -parabolic of  $F_4$ , hence  $H$  is in an  $A_1$ -parabolic of  $E_6$  (so that  $L'$  of type  $A_1$ ). But  $V_{27} \downarrow L' = L(1)^9 + k^9$  and so  $H$  is not  $GL(V_{27})$ -conjugate to (a subgroup of)  $L'$ , let alone  $E_6$ -conjugate. Now  $V_{27} \downarrow L' \cong V_{27}^* \downarrow L'$  and  $V_{27} \downarrow H \cong V_{27}^* \downarrow H$ . Since the  $E_7$ -module  $V_{56}$  has  $V_{56} \downarrow E_6 = V_{27} \oplus V_{27}^* + k^2$  we see  $H$  is also non- $E_7$ -cr.

To show it is also non- $E_8$ -cr, note that  $L(E_8) \downarrow E_7 = L(E_7) + L(T_1) + L(Q)^2$  where  $Q$  is the unipotent radical of an  $E_7$ -parabolic of  $E_8$ , with  $L(Q) \downarrow E_7 = V_{56} + k$ . Thus  $L(E_8) \downarrow H$  contains at least two submodules isomorphic to  $T(2)$  (contained in the two  $V_{56}$ s). On the other hand  $L(E_8) \downarrow L' = L(A_1 T_1) + k^6 + M^2$  where  $M$  is the restriction to  $L'$  of the Lie algebra of the unipotent radical of an  $A_1$ -parabolic. Using 5.1,  $M$  has composition factors with high weights 1 or 0, which must be semisimple since  $\text{Ext}_{A_1}^1(L(1), L(1)) = \text{Ext}_{A_1}^1(L(1), L(0)) = 0$ . In particular, while the direct summand  $L(A_1 T_1)$  is an indecomposable module  $T(2)$  for  $L'$ , it is the only one in  $L(E_8)$ ; for  $H$  there are at least two such (in  $L(Q)$ ). Thus  $H$  is also non- $E_8$ -cr.

$$(G, X, p) = (E_6, A_2, 3)$$

Let  $\tau$  denote a graph automorphism of  $G$  with induced action on the Dynkin diagram for  $G$ . If  $G_\tau$  denotes the fixed points of  $\tau$  in  $G$ , we have  $G_\tau \cong F_4$  such that the root groups corresponding to simple short roots are contained in the subsystem (of type  $A_2 A_2$ ) determined by the nodes in the Dynkin diagram of  $G$  on which  $\tau$  acts non trivially. Thus  $H$  is contained in  $A_2 \tilde{A}_2 \leq F_4$  by  $x \mapsto (x, x)$ . It is shown in [Ste12, 4.4.1, 4.4.2] that this subgroup is in a  $B_3$ -parabolic of  $F_4$  with  $V_7 \downarrow \bar{H} = L_{A_2}(11)$ .

In [Ste12, 5.1] the restrictions of the  $F_4$ -module  $V_{26} = V(0001) \cong 0001/0000$  to  $H$  and  $\bar{H}$  is calculated. Using this together with  $V_{27} \downarrow F_4 = T(0001) = 0000/0001/0000$  we see that  $V_{27} \downarrow \bar{H}$  cannot be the same as  $V_{27} \downarrow H$ : the former is an extension by the trivial module of  $V_{26} \downarrow \bar{H} = 11^3 + 00^5$  where the resulting module is self-dual, so must be  $11^3 + 00^6$  whereas the latter is forced to be  $T(11)^3$ . By a similar argument as before, we also get that this subgroup is non- $E_7$ -cr and non- $E_8$ -cr.

We give an example of a subgroup not arising from a non- $F_4$ -cr subgroup (these being found in [Ste12]):

$$(G, X, p) = (E_6, A_1, 5).$$

The module  $T(8) = L(0)/L(8)/L(0)$  is a direct summand of the 25-dimensional module  $L(4) \otimes L(4) = T(4) \otimes T(4)$  by 2.1. The two tensor factors here admit orthogonal forms, so the tensor product does too. Hence we get a subgroup of type  $SL_2$  in  $GL_{25}$  which is actually contained in  $SO_{25}$ . Indeed as the 10-dimensional direct factor  $T(8)$  is the unique such, the duality must preserve this factor. Hence we get an  $A_1 \leq SO_{10} \times SO_{15}$  and so projecting to the first orthogonal group, we get  $H \leq SO_{10}$  with  $V_{10}|H = T(8)$ .

Now, by 4.8 we have that this subgroup is in a parabolic of  $SO_{10}$ . Considering dimensions of composition factors of Levi subgroups of  $D_5$  acting on the natural module shows that  $H$  must in fact be in a  $D_4$ -parabolic of  $D_5$  with  $\bar{H}$  being  $D_4$ -irreducible and  $V_8 \downarrow \bar{H} = L(8)$ . By e.g. [Ste12, 5.1] we can calculate  $V_{27} \downarrow D_4 = L(\omega_1) + L(\omega_3) + L(\omega_4) + k^3$ . We wish to restrict this further to get  $V_{27}|\bar{H}$  and  $V_{27}|H$ . Note that since  $L(8) \cong L(3) \otimes L(1)^{[1]}$ , we have  $\bar{H} \leq Sp_4 \times Sp_2 \leq D_4$ . Let  $\bar{H}'$  (resp.  $\bar{H}''$ ) denote the projection of the  $\bar{H}$  in the first (resp. second) factor. Taking a graph automorphism, we can consider  $SL_4$  as type  $D_3$  corresponding to nodes 2, 3 and 4 of the Dynkin diagram. Then we have  $L_{D_4}(\omega_1)|SL_4 = L(010) + k^2$ , thus  $L_{D_4}(\omega_1)|\bar{H}' = \bigwedge^2(L(3)) + k^2 =$

$L(4) + k^3$ , with  $L_{D_4}(\omega_1)|\bar{H} = L(4) + L(1)^{[1]} + k$  or  $L_{D_4}(\omega_1)|\bar{H} = L(4) + L(2)^{[1]}$ . As  $\bar{H}$  is  $D_4$ -ir, it must be the latter, since  $L(1)^{[1]}$  carries a symplectic form. Also from [LS96, 2.7] one sees that  $L_{D_4}(\omega_3) \downarrow SL_4 \cong L_{D_4}(\omega_4) \downarrow SL_4 = L(100) + L(001)$  and so  $L_{D_4}(\omega_3) \cong L_{D_4}(\omega_4)|\bar{H}' = L(3)^2$ . Thus  $L_{D_4}(\omega_3) \cong L_{D_4}(\omega_4)|\bar{H} = L(3) \otimes L(1)^{[1]} = L(8)$ .

Finally we conclude that  $V_{27} \downarrow \bar{H} = L(8)^2 + L(4) + L(2)^{[1]} + 0^3$ . In particular,  $\bar{H}$  acts semisimply. On the other hand  $V_{27} \downarrow D_5 = L(\omega_1) + L(\omega_5) + k$  where  $L(\omega_5)$ . But  $H$  does not act semisimply on  $V_{10}$ . So  $\bar{H}$  is not  $GL(V_{27})$ -conjugate to  $H$ , so neither is it  $E_6$ -conjugate to  $H$ .

The remaining cases where  $X = A_1$  are similar.

Let us now vouch for the existence of the subgroup asserted in case

$$(G, X, p) = (E_8, C_3, 3).$$

First observe that since the natural module  $L(100)$  for  $Sp_6$  admits a symplectic form, the tensor square  $M = L(100) \otimes L(100)$  admits an orthogonal form, with composition factors  $L(200)|L(010)|L(000)^2$ . Since  $L(100)$  is a tilting module, so is  $M$ ; and since  $L(200) = V(200) = T(200)$ , while [Lüb01] gives  $V(010) = L(010)/L(000)$  we must have  $M \cong L(200) + T(010)$ . Duality preserves these factors, so the 15-dimensional module  $T(010)$  is orthogonal for  $Sp_6$ . Thus we have a subgroup  $Sp_6 \leq SO_{15} \leq SO_{16}$  obviously in a  $D_7$ -parabolic of this  $D_8$ .

$$H = E(C_4, E_7, 2)$$

is discussed in [LST96, 2.7, Proof]; there it is shown to be in an  $E_6$ -parabolic and not conjugate to its image  $\bar{H} \cong C_4 \leq F_4 \leq E_6 = L'$ . We need to show that this subgroup is also non- $E_8$ -cr. For this, restriction of  $L(E_8)$  to an  $E_6$  Levi gives  $L(E_8)|_{E_6} \cong L(E_6 T_2) + L(Q) + L(Q^-)$ , with  $L(Q)$  having composition factors  $k$ ,  $L(\omega_1)$  or  $L(\omega_6)$  by 5.1. We have  $L(\omega_6)|\bar{H} = L(\omega_1)|\bar{H} = L(0100) + k$ , and  $L(E_6 T_2) \cong L(E_6) + k^2$  has dimension 80. On the other hand,  $L(E_8)|_{E_7} = L(E_7 T_1) + L(R) + L(R^-)$  for  $R$  the unipotent radical of an  $E_7$ -parabolic. By 5.1  $L(R)|_{E_7} = V_{56} + k$ . But  $V_{56}|_{A_7} = L(\lambda_2) + L(\lambda_6)$  from [LS96, 2.7]. Thus  $V_{56}|_H = \bigwedge^2(L(1000)) + (\bigwedge^2(L(1000)))^* = T(0100)^2$ .<sup>4</sup> In particular there are 4 direct factors in  $L(E_8)|_H$  which are isomorphic to the 28-dimensional module  $T(0100)$ . However we found above that there are none in the submodule  $(L(Q) + L(Q^-))|\bar{H}$  of  $L(E_8)|\bar{H}$ , so if  $H$  were conjugate to  $\bar{H}$ , one would have to find these 4 direct factors  $T(0100)$  inside  $L(E_6 T_1)$ ; but the dimension of the latter is  $79 < 4 \times 28 = 112$ .

There is one further case where we could not give a nice embedding as we have done above. Let

$$H = (E_7, G_2, 7).$$

We first indicate how to see the existence of this subgroup then show that it cannot have any proper reductive overgroup. By [LS04], when  $p = 7$ ,  $F_4$  has a maximal subgroup of type  $G_2$ . Set  $\bar{H}$  to be this subgroup and regard  $\bar{H}$  as subgroup of a Levi subgroup of an  $E_6$ -parabolic; note that  $\bar{H}$  is  $E_6$ -irreducible. By 4.9 one has  $V_{27}|\bar{H} = L(20) + k$ . Now, using [Lüb01], one has, when  $p = 7$  that  $V(20)$  is uniserial with composition factors  $20|00$ . Thus  $H^1(\bar{H}, L(20)) = H^0(\bar{H}, H^0(20)/L(20)) = k$ . Now

<sup>4</sup>One way to see this is to note that  $T = L(1000) \otimes L(1000)$  is a tilting module, whose character can be decomposed to yield composition factors  $L(2000)|L(0100)^2|L(0000)^4$ . Now, one can use Doty's Weyl group package for GAP to see that  $V_{C_4}(2000)$  is uniserial with successive factors  $L(2000)|L(0000)|L(0100)|L(0000)$  and  $V_{C_4}(0100)$  is uniserial with successive factors  $L(0100)|L(0000)$ . Thus  $T(2000)$  is uniserial with successive factors  $L(0000)|L(0100)|L(0000)|L(2000)|L(0000)|L(0100)|L(0000)$  (it is clear that it has both a Weyl- and dual Weyl-filtration). So  $T = T(2000)$  and indecomposable. But  $\bigwedge^2(1000)$  is a submodule of  $T$ ; dimension considerations imply that it consists of the last three factors. But  $T(0100) = L(0000)|L(0100)|L(0000)$  so the claim follows.

$Q|L' \cong V_{27}$  or  $V_{27}^*$  so one has  $H^1(\bar{H}, Q) = k$ . Now by [Ste12, 3.2.15] it follows that there is a non- $G$ -cr subgroup  $H$ , which is a complement to  $Q$  in  $\bar{H}Q$ .

Suppose  $H$  had a proper reductive overgroup in  $G$ . Then by 4.9 it would have to lie in a subsystem subgroup of type  $A_7$ . Also it cannot lie in any parabolic subgroup of  $A_7$  since then  $\bar{H}$  would not be  $E_6$ -irreducible. Checking [Lüb01] one sees that there are no irreducible 8-dimensional representations of  $H \cong G_2$ . This is a contradiction. Thus  $H$  has no proper reductive overgroup in  $G$  as required.

The remaining cases are all similar and easier. This completes the proof of Theorem 1.

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